

Applying Risk Theory to Game Theory

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Abstract

The *Minimax Theorem* is the most recognized theorem for determining strategies in a two person zero-sum game. Other common strategies exist such as the *maximax principle* and *minimize the maximum regret principle*. All these strategies follow the Von Neumann and Morgenstern linearity axiom which states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x)=ax+b$, $a>0$ without changing the information they convey. This paper describes risk-averse strategies where the linearity axiom may not hold. With connections to gambling theory, there is evidence to show why it can be optimal for the favorable player to adopt risk-averse strategies. This reasoning can then be applied to a two person zero-sum game arbitration process to determine an alternative outcome to the solution given by strategies under the Minimax Theorem. Risk analysis is used to show why the uncooperative solution in the Prisoner's Dilemma can be a "reasonable" outcome to the game, even though the solution is *non-Pareto optimal*, and why cooperation may be implicitly forced in the game. Logical reasoning is given to show why maximin strategies in two person nonzero-sum games could be considered as a Nash Equilibria. A risk-averse status quo is devised for the Nash Arbitration scheme as an alternative to the maximin and threat status quo solutions. The analysis and results given in this paper show that it can be "optimal" for the favorable player to accept less than the amount given by maximizing expectation, due to the risks involved and the possibility of having a negative payout.

1. Introduction

The mathematical foundations of game theory as outlined in Von Neumann and Morgenstern (1944) is built around maximizing expectations. This is shown in a game theory axiom which states that numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x)=ax+b$, $a>0$ without changing the information they convey. Whilst a player may choose to maximize expectations, a player may be concerned about obtaining a negative payout or their maximum possible loss (MPL) for the game. This is likely to be more of a concern when the MPL is a negative payout and to reduce the MPL occurring, a player may choose strategies accordingly. In effect, the maximal expectation is reduced by minimizing the probability of obtaining the MPL i.e. risk-averse strategies.

Risk-averse strategies have foundations in gambling theory when a favorable game exists. Friedman (1980) and Schlesinger (2004) outline risk-averse strategies in blackjack where the objective is to minimize the probability of losing one's bankroll. These risk-averse strategies give a positive expectation which is less than the expectation when the objective is to maximize the payout by assuming a "large" bankroll.

This paper begins with section 2 which outlines risk-averse strategies for the favorable player in a 2 x 2 zero-sum game. The risk of ruin and the Kelly Criterion methods are applied to game theory based on

their applicability to gambling theory. Section 3 applies risk-averse strategies to a 2 x 2 nonzero-sum game. The Prisoner's Dilemma game is given to demonstrate why players may choose to cooperate or not to cooperate, and demonstrates that an uncooperative solution can be a "reasonable" outcome to the game, even though the solution is *non-Pareto optimal*. Logical reasoning is given to show why maximin strategies in nonzero-sum games could be considered as a Nash Equilibria, which are used to establish "optimal" playing strategies for both players. Section 3 concludes with devising a risk-averse status quo for the Nash Arbitration scheme.

2. Two person zero-sum game

2.1 Minimax Theorem

Consider the following 2 x 2 game (call it Game 1)

		P2	
		A	B
P1	A	2/3	-1
	B	-1/3	1

Game 1

The most common solution to this type of game is using the Minimax Theorem to obtain player strategies. This gives mixed strategies as P1: 4/9A, 5/9B and P2: 2/3A, 1/3B. The value of the game is 1/9.

Table 1 represents Game 1 in a form where both players apply strategies from the Minimax Theorem. Outcome AA refers to P1 using strategy A and P2 using strategy A. Outcomes AB, BA and BB are obtained similarly. The value of the game is given as 1/9 as expected. This representation is typically given for a casino game.

Outcome	Payout	Probability	Expected Payout
AA	2/3	$4/9 * 2/3 = 8/27$	$2/3 * 8/27 = 16/81$
AB	-1	$4/9 * 1/3 = 4/27$	$-1 * 4/27 = -4/27$
BA	-1/3	$5/9 * 2/3 = 10/27$	$-1/3 * 10/27 = -10/81$
BB	1	$5/9 * 1/3 = 5/27$	$1 * 5/27 = 5/27$
		1	1/9

Table 1: Probabilities and expected payouts for Game 1 with both players using strategies under the Minimax Theorem

Table 2 is an extension of Table 1 which is used to obtain moments, which can then be used to obtain cumulants and common distributional characteristics as follows:

Mean = 0.111

Standard Deviation = 0.703

Coefficient of Variation = 6.325
 Coefficient of Skewness = -0.158
 Coefficient of Excess Kurtosis = -1.425

These distributional characteristics (other than the mean) provide extra information to the possible payouts in a two person zero-sum game.

Outcome	Payout	Probability	1 st Moment	2 nd Moment	3 rd Moment	4 th Moment
AA	2/3	8/27	0.198	0.132	0.088	0.059
AB	-1	4/27	-0.148	0.148	-0.148	0.148
BA	-1/3	10/27	-0.123	0.041	-0.014	0.005
BB	1	5/27	0.185	0.185	0.185	0.185
			0.111	0.506	0.111	0.396

Table 2: The first four moments for Game 1 with both players using strategies under the Minimax Theorem

2.2 Existing Strategies

Commonly used strategies other than the strategies determined by the Minimax Theorem are as follows as documented in Straffin (1993): Suppose P1 recognizes that P2 will play mixed strategies as 0.8A, 0.2B. Then using the Expected Value Principle, P1 would use the fixed strategy of A since the reward for the game is 1/3. Players may want to be cautious and Wald's method for P1 is to write down the minimum entry in each row and choose the row with the largest minimum. This would mean P1 would use a fixed strategy of B. An analog for P2 would be to use the fixed strategy of A. As Wald's *maximin* strategy is looking at the worst that will happen, the corresponding principle for optimists would be the *maximax* principle. This would mean P1 would use a fixed strategy of B and P2 would use a fixed strategy of B. Hurwicz combined these two approaches by choosing a "coefficient of optimism" α between 0 and 1. For each row compute $\alpha(\text{row maximum}) + (1 - \alpha)(\text{row minimum})$. Choose the row for which this weighted average is the highest. For example, suppose we choose $\alpha = 0.8$, then A: $0.8(2/3) + 0.2(-1) = 1/3$ and B: $0.8(-1/3) + 0.2(1) = -1/15$. Hence P1 would choose strategy A. Savage's method involves regret by writing down the largest entry in each row. Choose the row for which this largest entry is smallest. This would mean P1 would choose the fixed strategy of A and P2 would choose the fixed strategy of A.

2.3 New Strategies: Risk-averse

Von Neumann and Morgenstern (1944) developed the framework for modern utility theory by assigning numbers to outcomes in a way which reflect an actor's preference. The fundamental ideas of this theory affect game theory. It is stated that for a mixed strategy game solution to be meaningful, the numbers in the game matrix must be cardinal utilities and can be transformed by any positive linear function $f(x)=ax+b$, $a>0$ without changing the information they convey. All the strategies in section 2.2 and the

strategies determined by the Minimax Theorem conform to this property. However, this property does not take into account risk and whether a player (presumably the player with an expected positive payout) would reduce the expected payout in order to reduce risk and minimize the probability of obtaining the maximum possible loss. Consider the example given in Stahl (1999). Suppose the game (call it Game 2) is played 10 times and P1 uses strategies from the Minimax Theorem of P1: A=0.2, B=0.8. If by chance in those 10 trials, P1 selects the first row on the first, fifth and ninth plays only and P2 selects the second column on those same plays only, then P1 ends up with a total payout of 5, which is much less than the total of 20, P1 would have been sure to win had they always selected the second row.

		P2	
		A	B
P1	A	5	-3
	B	2	4

Game 2

Von Neumann and Morgenstern (1944) were aware of the relativistic value of the strategies used from the Minimax Theorem. In their words:

All this may be summed up by saying that while our good strategies are perfect from the defensive point of view, they will (in general) not get the maximum out of the opponent's (possible) mistakes, - i.e. they are not calculated for the offensive. It should be remembered, however, that our deductions of 17.8 are nevertheless cogent; i.e. a theory of the offensive, in this sense, is not possible without essentially new ideas.

Definition 1 describes risk-averse strategies for a 2 x 2 zero-sum game which provides a new framework from the theory developed by Von Neumann and Morgenstern (1944).

Definition 1: Consider a 2 x 2 zero-sum game where at least one of the payouts is positive and at least one of the payouts is negative. The value v of the game under the Minimax Theorem is either positive, negative or zero. Risk-averse strategies can be obtained when v is positive such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the maximum possible loss (MPL) for P1 in the game is reduced when compared to the strategies under the Minimax Theorem. Risk-averse strategies for P2 when v is negative follow.

Using Definition 1, the risk-averse solution for P1 to Game 1 is obtained as P1: $1/3 < A < 4/9$, $5/9 < B < 2/3$. These calculations were obtained by noting that P1 always obtains an expected positive payout regardless of the strategies used by P2. Table 3 represents the distributional characteristics for Game 1 with P1 and P2 using different strategies. Column two gives the characteristics for when both players are using strategies under the Minimax Theorem. Column three gives the characteristics for when P1 uses risk-averse strategies and P2 uses strategies from the Minimax Theorem. It is observed in column three that the maximum possible loss is reduced to agree with Definition 1, as well as the standard deviation,

coefficient of variation and coefficients of skewness and excess kurtosis. Column four gives the characteristics for when P1 uses risk-averse strategies and P2 uses strategy A=1, B=0. For this scenario the maximum possible loss is reduced as well as the standard deviation. The coefficient of skewness has become positive which is also an indicator of a strategy that is risk-averse.

Distributional Characteristic	P1: 4/9A, 5/9B P2: 2/3A, 1/3B	P1: 0.4A, 0.6B P2: 2/3A, 1/3B	P1: 0.4A, 0.6B P2: A=1, B=0
Mean	0.111	0.111	0.067
Standard Deviation	0.703	0.696	0.490
Coefficient of Variation	6.325	6.261	7.348
Coefficient of Skewness	-0.158	-0.095	0.408
Coefficient of Excess Kurtosis	-1.425	-1.424	-1.833
Maximum Possible Loss	0.148	0.133	0.000
Overall Loss	0.519	0.533	0.600

Table 3: Distributional characteristics of Game 1 for various strategies used by P1 and P2

Example 1: Using Game 1, suppose P1 is restricting the probability of the -1 payout to be 0.12 (given P2 uses mixed strategies obtained from the Minimax Theorem). Then $0.12 / 1/3 = 0.36$ and P1 should use the risk-averse strategy of P1: 0.36A, 0.64B. If P2 identified that P1 was deviating from the Minimax Theorem, then P2 could use strategy A for an expected payout for P2 of -0.027.

2.4 Risk of Ruin

A common problem that often arises in gambling is obtaining the probability of losing one's entire bankroll given a favorable game. For blackjack, the risk of ruin can be approximated by simple explicit formulas, and is reasonably accurate as the distribution of profits is roughly normally distributed. For games like video poker, Caribbean stud poker and certain advantageous slot machines, the distribution of profits are often highly right-skewed and therefore the approximation methods used in blackjack are inaccurate. The following recursive solution (which assumes independent trials) was derived by Evgeny Sorokin and posted on Arnold Snyder's Blackjack Forum Online <http://www.blackjackforumonline.com/content/VPRoR.htm>.

The equation is given as $R(1) = E[p_i * R(1)^{Z_i}]$

where:

R(1) is the risk of losing a 1 unit bankroll

Z_i is the return payoff for outcome i

p_i is the associated probability for Z_i

For the risk of ruin to be meaningful in a 2 x 2 zero-sum game, two of the payoffs must be negative and the other two must be positive. In the context of Game 1, suppose P1 has a bankroll of 3 units. With P1

and P2 playing strategies under the Minimax Theorem, the risk of ruin for P1 is obtained as 26.0877%. Suppose P1 increased strategy B to 0.613, then the risk of ruin (with P2 playing strategies under the Minimax theorem) is reduced to 24.87227%. If P2 then increased strategy A, the risk of ruin for P1 would only decrease as shown in table 4.

	P1: A = 0.387, B =0.613		P1: A = 0.386, B =0.614	
Probability	Risk of Ruin for P1	Expected Payout	Risk of Ruin for P1	Expected Payout
P2: A=2/3	24.87227%	0.11111	24.85068%	0.11111
P2: A=0.667	24.87197%	0.11105	24.85069%	0.11105
P2: A=0.668	24.87107%	0.11088	24.85070%	0.11088
P2: A=0.70	24.84051%	0.10537	24.85120%	0.10527

Table 4: Risk of ruin and expected payouts for Game 1 for various strategies used by P1 and P2

Suppose P1 increased strategy B to 0.614, then the risk of ruin (with P2 playing strategies under the Minimax theorem) is reduced to 24.85068%. If P2 then increased strategy A, the risk of ruin for P1 would only increase as shown in table 4. Therefore, there appears to be a strategy B value for P1 between 0.613 and 0.614 where the risk of ruin remains constant as P2 increases strategy A from the strategies obtained from the Minimax theorem. Similar results were obtained by assigning P1 a bankroll of 10 units. Therefore, it appears that the strategies for P1 to minimize the risk of ruin are independent of the size of the bankroll.

Suppose P1 uses strategies of 0.387A, 0.613B. The expected payout for P1 will be 0.11111 if P2 uses strategies under the Minimax Theorem and 0.05367 if P2 uses the strategy of A=1, B=0. The risk of ruin with a 3 unit bankroll will be 24.87227% if P2 uses strategies under the Minimax Theorem and 24.22377% if P2 uses the strategy of A=1, B=0. Note that the risk of ruin if both P1 and P2 use strategies under the Minimax Theorem is 26.0877%. Hence, P1 can reduce the risk of ruin by reducing the expected payout.

Suppose the payouts for each player in Game 1 was to be determined by an outside arbitrator. One obvious method is simply to use the value of the game given by the Minimax Theorem. For Game 1 this would be 1/9 to P1. However, as shown from above, there are benefits for the favorable player to be risk-averse to minimize the risk of ruin. From that example, P1 would use the strategies of A =0.387, B=0.613.

Suppose P2 increased strategy A to 0.689, then the risk of ruin (with P1 playing strategies under the Minimax theorem) is 24.87148%. If P1 then increased strategy B, the risk of ruin for P1 would only decrease as shown in table 5.

Suppose P2 increased strategy A to 0.690, then the risk of ruin (with P1 playing strategies under the Minimax theorem) is 24.81632%. If P1 then increased strategy B, the risk of ruin for P1 would only increase as shown in table 5.

	P2: A=0.689, B=0.311		P2: A=0.690, B=0.310	
Probability	Risk of Ruin for P1	Expected Payout	Risk of Ruin for P1	Expected Payout
P1: B=5/9	24.87148%	0.11111	24.81632%	0.11111
P1: B=0.556	24.87133%	0.11108	24.81657%	0.11108
P1: B=0.557	24.87099%	0.11101	24.81714%	0.11101
P1: B=0.60	24.85609%	0.10813	24.84242%	0.10800

Table 5: Risk of ruin and expected payouts for Game 1 for various strategies used by P1 and P2

This example and reasoning suggests that an alternative arbitration to account for risk is for P1 to obtain the value of 0.10726 from P2. This was obtained by P1 using strategies of A = 0.387A, 0.613 B and P2 using strategies of 0.689A, 0.311B. Note that the payout for P1 of 0.10726 is a reduction from the expected payout given by the Minimax Theorem of 0.11111.

Definition 2 is now given for risk-averse strategies in an $m \times n$ zero-sum game based on Definition 1 and the risk of ruin methodology presented in this section.

Definition 2: Consider an $m \times n$ zero-sum game. The value v of the game under the Minimax Theorem is either positive, negative or zero. Risk-averse strategies are given for (1): when v is positive and at least two of the payouts are negative, such that the expected payout for P1 is positive regardless of the strategies used by P2, and the risk of ruin for P1 in the game is reduced when compared to the strategies under the Minimax Theorem. Risk-averse strategies for P2 when v is negative follow. Risk-averse strategies are given for (2): when v is positive and only one of the payouts is negative, such that the expected payout for P1 is positive regardless of the strategies used by P2, and the probability of obtaining the negative payout for P1 in the game is reduced when compared to the strategies under the Minimax Theorem. Risk-averse strategies for P2 when v is negative follow.

2.5 Kelly Criterion

The Kelly Criterion is typically applied to favorable casino games to maximize the long term growth of the bankroll. The Kelly criterion for when multiple outcomes exist is given as follows:

Consider a game with m possible discrete finite mixed outcomes. Suppose the profit for a unit wager for outcome i is k_i with probability p_i for $1 \leq i \leq m$, where at least one outcome is negative and at least one outcome is positive. Then if $\sum_{i=1}^m k_i p_i > 0$ a winning strategy exists, and the maximum growth of the bank is attained when the proportion of the bank bet at each turn, b , is the smallest positive root of

$$\sum_{i=1}^m \frac{k_i p_i}{1+k_i b} = 0$$

The Kelly criterion is applied to Game 1 to demonstrate why the favorable player may consider risk-averse strategies. The value of b with both P1 and P2 using strategies under the Minimax Theorem is

obtained as 0.222. Therefore P1 (the favorable player) should wager an amount of $0.222 \times$ current bankroll to maximize the long term growth of the bank. Since the amount that P1 can bet is fixed at 1 unit, the decision on whether to apply strategies under the Minimax Theorem or risk-averse strategies depends on the size of the bankroll. Using Solver in Excel, Table 6 shows that P1 would need a bankroll of at least 4.51 units to avoid over betting by using strategies under the Minimax Theorem. By using risk-averse strategies, P1's bankroll can be less than 4.51 as given in Table 6.

Strategy	Bankroll (units)
P1: A=4/9, B=5/9	4.51
P1: A=0.4, B=0.6	4.38
P1: A=0.35, B=0.65	4.23

Table 6: Bankroll requirements when using strategies under the Kelly Criterion

The Kelly Criterion could also be used to determine arbitration based on the size of the bankroll for the favorable player. For example, if P1 had a bankroll of 4.38 units, then the risk-averse strategies given from Table 6 are A=0.4, B=0.6. With P2 playing strategies as A=1, B=0 the negotiated amount to P1 is 0.067.

3. Two Person Nonzero-sum Game

3.1 Existing Strategies: Nash Equilibria and Maximin

When a player adopts strategies under the Minimax Theorem in an $m \times n$ zero-sum game, they are guaranteed two important outcomes:

- 1) If the opponent P2 deviates from their strategies under the Minimax Theorem, then the player's P1 expected payout is at least v (v is identified as the security level)
- 2) The player cannot achieve a better expected payout by deviating from their strategies under the Minimax Theorem (identified as the No Regret policy)

In an $m \times n$ nonzero-sum game there is no general solution to guarantee the above two outcomes. The most recognized strategies is using *Nash Equilibria* strategies and has been identified as a No Regret Policy. The maximin strategies are also well known and are interpreted as security levels in that each player is guaranteed to obtain as least a certain expected payout regardless of the strategies used by the opponent.

The following is representative of a 2×2 nonzero-sum game with payouts to P1 and P2.

		P2	
		A	B
P1	A	a_1, a_2	b_1, b_2
	B	c_1, c_2	d_1, d_2

If a mixed maximin exists, then the maximin strategies are obtained as:

$$P1: A = (d_1 - c_1) / (a_1 + d_1 - c_1 - b_1), B = (a_1 - b_1) / (a_1 + d_1 - c_1 - b_1)$$

$$P2: A = (d_2 - b_2) / (a_2 + d_2 - c_2 - b_2), B = (a_2 - c_2) / (a_2 + d_2 - c_2 - b_2)$$

Let e_{MM1} and e_{MM2} represent the expected payouts for P1 and P2 respectively when both P1 and P2 adopt mixed maximin strategies. It can be shown that:

$$e_{MM1} = (a_1d_1d_2 - a_1d_1b_2 - b_1c_1a_2 + b_1c_1c_2 - c_1b_1d_2 + c_1b_1b_2 + d_1a_1a_2 - d_1a_1c_2) / x$$

$$e_{MM2} = (a_2d_1d_2 - a_2c_1d_2 - b_2d_1c_2 + b_2c_1c_2 - c_2a_1b_2 + c_2b_1b_2 + d_2a_1a_2 - d_2b_1a_2) / x$$

$$\text{where } x = (a_1 + d_1 - c_1 - b_1)(a_2 + d_2 - c_2 - b_2)$$

If a mixed *Nash Equilibria* exists, then the *Nash Equilibria* strategies are obtained as:

$$P1: A = (d_2 - c_2) / (a_2 + d_2 - c_2 - b_2), B = (a_2 - b_2) / (a_2 + d_2 - c_2 - b_2)$$

$$P2: A = (d_1 - b_1) / (a_1 + d_1 - c_1 - b_1), B = (a_1 - c_1) / (a_1 + d_1 - c_1 - b_1)$$

Let e_{NN1} and e_{NN2} represent the expected payouts for P1 and P2 respectively when both P1 and P2 adopt mixed *Nash Equilibria* strategies. It can be shown that:

$$e_{NN1} = (a_1d_2d_1 - a_1c_2d_1 - b_1d_2c_1 + b_1c_2c_1 - c_1a_2b_1 + c_1b_2b_1 + d_1a_2a_1 - d_1b_2a_1) / x$$

$$e_{NN2} = (a_2d_2d_1 - a_2d_2b_1 - b_2c_2a_1 + b_2c_2c_1 - c_2b_2d_1 + c_2b_2b_1 + d_2a_2a_1 - d_2a_2c_1) / x$$

It can be verified from above that $e_{MM1} = e_{NN1}$ and $e_{MM2} = e_{NN2}$

Let e_{MN1} and e_{MN2} represent the expected payouts for P1 and P2 respectively when P1 adopts mixed maximin strategies (assuming it exists) and P2 adopts mixed *Nash Equilibria* strategies (assuming it exists). It can be shown that:

$$e_{MN1} = (a_1d_1d_1 - a_1c_1d_1 - b_1d_1c_1 + b_1c_1c_1 - c_1a_1b_1 + c_1b_1b_1 + d_1a_1a_1 - d_1b_1a_1) / y$$

$$e_{MN2} = (a_2d_1d_1 - a_2d_1b_1 - a_2c_1d_1 + a_2c_1b_1 + b_2d_1a_1 - b_2d_1c_1 - b_2c_1a_1 + b_2c_1c_1 + c_2a_1d_1 - c_2a_1b_1 - c_2b_1d_1 + c_2b_1b_1 + d_2a_1a_1 - d_2a_1c_1 - d_2b_1a_1 + d_2b_1c_1) / y$$

$$\text{where } y = (a_1 + d_1 - c_1 - b_1)^2$$

It can be verified from above that $e_{MN1} = e_{MM1}$ and $e_{MN2} \geq e_{MM2}$

This justifies why the *Nash Equilibria* is considered as the No Regret Policy as the same expected payouts are obtained regardless of whether the other player plays *Nash Equilibria* or maximin strategies. By adopting maximin strategies, the other player could benefit by increasing their expected payout

(compared to both players adopting Nash Equilibria) by playing *Nash Equilibria* strategies. However, maximin strategies have the powerful property in establishing a security level. By using *Nash Equilibria* strategies, the expected payout could be less than the expected payout by using maximin strategies. This could be of particular importance to a player that can guarantee a positive expected payout by using maximin strategies. This concept is demonstrated in section 3.2.

3.2 New Strategies: Risk-averse

Nonzero-sum games have the added complexity from zero-sum games in that cooperation may require communication between the players. Firstly, we will consider the situation when cooperation is not required.

Consider the following game (call in Game 3)

		P2	
		A	B
P1	A	2/3, 3	-1, -3
	B	-1/3, -2	1, -1

Game 3

There are three *Nash Equilibria* strategies calculated as:

- 1) P1: $A=1/7$, $B=6/7$ and P2: $A=2/3$, $B=1/3$. The expected payout is given as $(1/9, -1.29)$
- 2) P1: $A=0$, $B=1$ and P2: $A=0$, $B=1$. The payout is given as $(1, -1)$
- 3) P1: $A=1$, $B=0$ and P2: $A=1$, $B=0$. The payout is given as $(2/3, 3)$

The maximin strategies are obtained by P1: $A=4/9$, $B=5/9$ and P2: $A=2/7$, $B=5/7$. The expected payout is given as $(1/9, -1.29)$.

In a two person zero-sum game from section 2.3, risk-averse strategies were developed based on the favorable player and minimizing the maximum possible loss. Risk-averse strategies utilize the fact that the favorable player can obtain a positive payout regardless of the strategies used by the other player. To ensure this is the case in a two person nonzero-sum game, maximin strategies are adopted as oppose to *Nash Equilibria* strategies. Referring to Game 3, P1 will always generate a positive expected payout regardless of the strategies used by P2 if P1: $1/3 < A < 0.5$, $0.5 < B < 2/3$. The maximin strategies are P1: $A=4/9$, $B=5/9$. Therefore, risk-averse strategies are P1: $1/3 < A < 4/9$, $5/9 < B < 2/3$. The risk-averse strategies agree with Game 1, since the payouts for P1 in Game 3 are the same as the payouts given in Game 1. Note that playing the *Nash Equilibria* mixed strategy of P1: $A=1/7$, $B=6/7$ could result in a negative expected payout if P2 plays strategy A. For this reason, the maximin strategy could also be identified as a No Regret Policy and by relaxing the linearity axiom, maximin strategies could also be considered as *Nash Equilibria* strategies.

Definition 3 describes risk-averse strategies for 2 x 2 non-zero sum games when cooperation is not required.

Definition 3: Consider a 2 x 2 nonzero-sum game. The expected payout of the game for each player is either positive, negative or zero when applying maximin strategies. When the expected payout for a player/s is positive, risk-averse strategies can be obtained without cooperation such that the probability of obtaining the MPL (assumed to be negative) is reduced when compared to maximin strategies.

Cooperation may be required for two reasons:

- 1) To increase the expected payout even though a positive expected payout is guaranteed with maximin strategies
- 2) For a player to obtain a positive payout, as the expected payout is negative with maximin strategies

Cooperation will now be demonstrated with the Prisoner's Dilemma.

3.3 Prisoner's Dilemma

Consider the Prisoner's Dilemma game below (call it Game 4)

		P2	
		A	B
P1	A	(0,0)	(-2,1)
	B	(1,-2)	(-1,-1)

Game 4

P1 B dominates P1 A and P2 B dominates P2 A, so there is a unique equilibrium (which is also the maximin strategy) at BB. However, BB is non-Pareto optimal since both P1 and P2 would be better off at AA. The following is recognized as the Pareto Principle.

Pareto Principle: To be acceptable as a solution to a game, an outcome should be Pareto optimal.

The general form of Prisoner's Dilemma is given as follows:

Conditions: $T > R > U > S$ and $R \geq (S+T)/2$

		P2	
		A	B
P1	A	(R,R)	(S,T)
	B	(T,S)	(U,U)

The strategies for P1 and P2 can be defined in terms of risk. The best solution is for both players to play strategy A for a payout of R units. The outcome with the least amount of risk is for both players to play strategy B for a payout of U units. A player can increase their expected payout by playing strategy A as oppose to strategy B, but runs the risk of a payout of only S units.

The following game (call it Game 5) is also a Prisoner's Dilemma

		P2	
		A	B
P1	A	(2,2)	(-1,3)
	B	(3,-1)	(1,1)

Game 5

There is a unique *Nash Equilibria* (which is also the maximin strategy) at BB which gives both players a payout of 1 unit. From Definition 3, there are no risk-averse strategies since playing strategy A will only increase the probability of obtaining the MPL of -1.

If P1 plays strategy B, then P1 is guaranteed a positive payout of at least 1 unit regardless of the strategies used by P2. Whereas playing strategy A could give a player a payout of -1, even though 2 units are possible if both players cooperate. As long as strategy A for P1 is < 0.5 , then the expected payout will be positive regardless of the strategies used by P2. Therefore, cooperation can increase the payout even though a positive payout is guaranteed with maximin strategies and suitable strategies for P1 (and also P2) are $0 \leq A < 0.5$, $0.5 < B \leq 1$.

Based on this reasoning a "reasonable" solution to a game does not have to be Pareto optimal, which contradicts the Pareto Principle.

Consider the following game (call it Game 6)

		P2	
		A	B
P1	A	(1,1)	(-2,2)
	B	(2,-2)	(-1,-1)

Game 6

There is a unique *Nash Equilibria* (which is also the maximin strategy) at BB which gives both players an expected payout of -1 unit. Definition 3 fails since playing strategy A to obtain a positive payout will only increase the probability of obtaining the MPL of -2.

If both P1 and P2 play strategy B, then they will both end up with a negative payoff of -1. If both P1 and P2 play strategy A, then they will both end up with a positive payoff of +1. If P2 plays strategy B, then P1 will always end up with a negative payout regardless of the strategy P1 uses. If P1 plays strategy B, then P2 will always end up with a negative payout regardless of the strategy P2 uses. Therefore cooperation is implicitly forced in Game 6 in order for both players to obtain a positive payout. We would like to know the strategy A probability for P1 such that the expected payout for P2 is 0 when P2 plays strategy A with probability 1. This probability is obtained as $2/3$. Therefore cooperation is required to obtain a

positive payout, as the payout is negative with maximin strategies and suitable strategies for P1 (and also P2) are $2/3 < A \leq 1$, $0 \leq B < 1/3$.

Based on Definition 3 and the methodology given in this section, the following rules are given for determining strategies for both players in an $m \times n$ nonzero-sum game.

1) When the expected payout for a player/s is positive when applying maximin strategies, either play maximin strategies or risk-averse strategies such that the probability of obtaining the MPL (assumed to be negative) is reduced when compared to maximin strategies.

2) When the expected payout for a player/s is negative when applying maximin strategies, then play mixed *Nash Equilibria* strategies if one exists otherwise a *Nash Equilibria* strategy.

3) If the solution to the game if both players were to use maximin strategies is non-Pareto optimal, then the favorable player/s may consider to increase the expected payout by adopting strategies that may not be risk-averse.

4) When the expected payout for a player/s is negative when maximin strategies are applied by both players, then this player/s may consider increasing the expected payout by adopting alternate strategies.

3.4 Nash Arbitration

Consider the following game (call it Game 7), as given in Thomas (1984).

		P2	
		A	B
P1	A	(1,2)	(8,3)
	B	(4,4)	(2,1)

Game 7

The maximin status quo point is obtained as $(3\frac{1}{3}, 2\frac{1}{2})$ with a bargaining solution of $(6\frac{2}{3}, 3\frac{1}{3})$. The threat status quo point is obtained as (1,2) and the threat solution is $(6\frac{1}{2}, 3\frac{5}{6})$. Notice that the threat solution is slightly more in P2's favour.

Consider the linear transformation of Game 7 by subtracting 3 from each payout as given in Game 8.

		P2	
		A	B
P1	A	(-2,-1)	(5,0)
	B	(1,1)	(-1,-2)

Game 8

The maximin status quo for Game 8 is obtained as $(\frac{1}{3}, -\frac{1}{2})$ with a bargaining solution of $(3\frac{2}{3}, \frac{1}{3})$. The threat status quo is obtained as $(-2, -1)$ and the threat solution is $(3\frac{1}{2}, \frac{3}{8})$. P1 could be considered as the favorable player with a positive expected payout using maximin strategies and P2 could be considered as the weaker player with a negative expected payout using maximin strategies.

Suppose Game 8 was to be played simultaneously for both players to obtain a payout. As identified from sections 3.2 and 3.3, P1 may prefer maximin or risk-averse strategies to guarantee an expected positive payout. P2 cannot guarantee an expected positive payout and may prefer to use strategies that maximize their expected payout by assuming that P1 is using strategies to guarantee an expected positive payout. Table 7 represents the payouts for different strategies that may be used by P2. It can be observed that by P2 playing strategies of $A=1, B=0$; (assuming P1 is using maximin strategies) P2 obtains a positive expected payout that is greater than the expected payout for P1.

P1 Strategy	P2 Strategy	P1 Payout	P2 Payout
Maximin (A=2/9, B=7/9)	Maximin (A=1/2, B=1/2)	$\frac{1}{3}$	$-\frac{1}{2}$
Maximin (A=2/9, B=7/9)	Mixed Nash Equilibria (A =2/3, B=1/3)	$\frac{1}{3}$	$-\frac{4}{27}$
Maximin (A=2/9, B=7/9)	Pure Nash Equilibria (A=0, B=1)	$\frac{1}{3}$	$-1\frac{5}{9}$
Maximin (A=2/9, B=7/9)	Pure Nash Equilibria (A =1, B=0)	$\frac{1}{3}$	$\frac{5}{9}$

Table 7: Payouts for the different strategies used by P2 in relation to Game 8

It could be argued that P1 would be more inclined than P2 to obtain an arbitration process rather than playing the game simultaneously against P2 to obtain a payout. This is reflected by the observation that P1 could end up with a negative payout by playing the game simultaneously against P2, even though the expected payout for P1 is guaranteed to be positive. Therefore it appears logical that a maximin strategy is used for P1 to obtain the status quo for the Nash arbitration process. Based on the reasoning from section 3.1 it appears logical that the mixed *Nash Equilibria* (assuming one exists) is used for P2 to obtain the status quo for the Nash arbitration process. This risk-averse status quo is therefore $(\frac{1}{3}, -\frac{4}{27})$ and the Nash Arbitration with these status quo values is obtained as $(2\frac{26}{27}, \frac{55}{108})$. Table 8 gives a comparison of the three different strategies for obtaining the Nash Arbitration values. The threat solution is slightly more in P2's favour when compared to the maximin solution. This is a result of P2 having a stronger ability to threaten and not related to being the weaker player. The payouts could have been devised in a way such that the threat solution is slightly more in P1's (the favourable player) favour. The risk-averse solution however will be in favour of the weaker player when compared to the maximin strategy for a nonzero-sum game.

Strategy	Nash Arbitration
Maximin	(3.666, 0.333)
Threat	(3.500, 0.375)
Risk-averse	(2.963, 0.509)

Table 8: Nash Arbitration values for different strategies in relation to Game 7

4. Conclusions

This paper has described risk-averse strategies where the linearity axiom may not hold. With connections to gambling theory, there is evidence to show why it can be optimal for the favorable player to adopt risk-averse strategies. This reasoning was then applied to an arbitration process to determine an alternative outcome to the solution given by strategies under the Minimax Theorem. Risk analysis was used to show why the uncooperative solution in the Prisoner's Dilemma can be a "reasonable" outcome to the game, even though the solution is *non-Pareto optimal*, and why cooperation may be implicitly forced in the game. Logical reasoning was given to show why maximin strategies in nonzero-sum games could be considered as a Nash Equilibria, which were used to establish "optimal" playing strategies for both players. A risk-averse status quo was devised for the Nash Arbitration scheme as an alternative to the maximin and threat status quo solutions. The analysis and results given in this paper show that it can be "optimal" for the favorable player to accept less than the amount given by maximizing expectation, due to the risks involved and the possibility of having a negative payout.

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